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DISTRIBUTION OF THE NATURAL FREQUENCIES OF A THIN ELASTIC SHELL OF ARBITRARY OUTLINE

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An asymptotic formula for the distribution function of the natural frequencies of a thin elastic shell is proved. The formula is used to determine the frequency density under different assumptions relative to the shell geometry. Density curves are presented.

1. Formulation of the problem. Fundamental results. The determination of the frequencies of a thin elastic shell clamped at the boundary results in the following eigenvalue problem (see [1], p. 97, [2], p. 297):

$$\sum_{j=1}^{3}\left(\frac{h^2}{12}n_{ij}+l_{ij}\right)u_j=\lambda u_i \qquad (i=1,2,3) \tag{1.1}$$

$$u_1\big|_\Gamma=u_2\big|_\Gamma=u_3\big|_\Gamma=\frac{\partial u_3}{\partial \nu}\bigg|_\Gamma=0 \tag{1.2}$$

Here u_i are components of the displacement vector of a point on the shell middle surface, l_{ij} and n_{ij} are the differential operators

$$\begin{aligned}
l_{11}u_1 &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_1-\\
&\quad \frac{1-\sigma}{2}\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_1-(1-\sigma)\frac{u_1}{R_1R_2}\\
l_{12}u_2 &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_2+\frac{1-\sigma}{2}\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_2\\
l_{13}u_3 &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)u_3\right]-\frac{1-\sigma}{AR_2}\frac{\partial u_3}{\partial\alpha}\\
l_{21}u_1 &= -\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_1+\frac{1-\sigma}{2}\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_1\\
l_{22}u_2 &= -\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_2-\frac{1-\sigma}{2}\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_2-\frac{1-\sigma}{R_1R_2}u_2\\
l_{23}u_3 &= \frac{1}{B}\frac{\partial}{\partial\beta}\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)u_3\right]-\frac{1-\sigma}{BR_1}\frac{\partial u_3}{\partial\beta}\\
l_{31}u_1 &= -\frac{1}{AB}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\partial}{\partial\alpha}Bu_1+\frac{1-\sigma}{AB}\frac{\partial}{\partial\alpha}\left(\frac{B}{R_2}u_1\right)\\
l_{32}u_2 &= -\frac{1}{AB}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\partial}{\partial\beta}Au_2+\frac{1-\sigma}{AB}\frac{\partial}{\partial\beta}\left(\frac{A}{R_1}u_2\right)
\end{aligned} \tag{1.3}$$

$$l_{33}u_3 = \left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1R_2} + \frac{1}{R_2^2}\right)u_3$$

$$n_{33}u_3 = \frac{1}{AB}\left(\frac{1}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{\beta}\frac{\partial}{\partial\beta}\right)\frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right)u_3 + \frac{1}{AB}\nu_{33}u_3$$

where ν_{33} is some self-adjoint second order operator whose explicit form is not needed later. For brevity of the exposition we assume

$$n_{ij} = 0 \tag{1.4}$$

for all the remaining values of i, j (see [2], p. 299). Here h is a small parameter, the relative shell thickness, in (1.1) – (1.3), σ is the Poisson's ratio, $\lambda = (1-\sigma^2)\rho E^{-1}\omega^2$, where ρ is the density, E is the Young's modulus, ω is the vibration frequency, $R_1^{-1}(\alpha,\beta)$ and $R_2^{-1}(\alpha,\beta)$ are the principal curvatures of the middle surface which is assumed sufficiently smooth. The middle surface is referred to the lines of curvature, and $A(\alpha,\beta)$ and $B(\alpha,\beta)$ are coefficients of the first quadratic form

$$ds^2 = A^2d\alpha^2 + B^2d\beta^2, \quad A(\alpha,\beta), \quad B(\alpha,\beta) \geqslant a > 0$$

The problem (1.1), (1.2) is self-adjoint and has a positive discrete spectrum. Let $N(\lambda)$ denote the distribution function of the eigenvalues of this problem ($N(\lambda)$ equals the number of eigenvalues not exceeding a given λ). The following assertion is proved below.

Theorem. The asymptotic formula

$$N(\lambda) = \frac{\sqrt{3}}{4\pi^2 h}\left[\iint_g \left(\int_0^{2\pi} \operatorname{Re}\sqrt{\lambda - \Omega(\theta,\alpha,\beta)}\,d\theta\right) AB\,d\alpha\,d\beta + O(h^\gamma)\right] \tag{1.5}$$

is valid for fixed $\lambda > 0$ and $h \to 0$. Here

$$\Omega(\theta,\alpha,\beta) = (1-\sigma^2)\,[R_1^{-1}(\alpha,\beta)\sin^2\theta + R_2^{-1}(\alpha,\beta)\cos^2\theta]^2 \tag{1.6}$$

$$0 \leqslant \theta \leqslant 2\pi, \qquad (\alpha,\beta)\in g, \qquad \gamma = \text{const} > 0$$

and g is the domain of variation of the parameters, and the constant in the O-term is independent of $0 \leqslant \lambda \leqslant \lambda_0$ (*).

Note 1. It will be shown that (1.5) is conserved if any differential operators containing not more than two differentiations for i and $j \leqslant 2$ and not more than three differentiations for $i = 3$, $j = 1, 2$ ($j = 3$, $i = 1, 2$) under the single assumption that the problem (1.1), (1.2) remains self-adjoint, are taken as n_{ij} in (1.1).

Note 2. It will be shown that (1.5) is valid for any boundary conditions for which the system (1.1) is self-adjoint, and the corresponding quadratic functional agrees with the quadratic functional of the problem (1.1), (1.2).

Let us note that in the case of a shell whose principal curvatures are almost constant an expression for $N(\lambda)$ agreeing with the inner integral in (1.7) was found by Bolotin [3] (see also [4], pp. 11, 13). In the case of a shell of revolution a formula for the density, analogous to (1.5), has recently been established by Tovstik [5] under certain added constraints. The validity of (1.5) in the case of a shell of revolution has been proved rigorously [6] in the whole frequency range and the remainder term has been estimated.

*) See Sects. 4, 6 below relative to the estimate of the remainder term in (1.5).

The validity of (1.5) in the case of an arbitrary shell has been proved in a development of [6] by the method of overlapping cells [7]. Formula (1.5) is proved below by the Hilbert-Courant scheme [8]. The idea of cutoff functions is used to overcome the fundamental difficulty, which is to prove the agreement between the asymptotics in the clamped and free cells case. The plan pointed out at the end of the Gol'denveizer paper ([9], p. 914) realizes the theorem as a whole. A number of other results associated with the distribution of the natural frequencies is also presented.

2. Estimates for the quadratic functional. Henceforth, the notation

$$(u, v) = \iint_g u\bar{v}AB\, d\alpha\, d\beta \tag{2.1}$$

is used for the scalar product of two functions $u(\alpha, \beta)$ and $v(\alpha, \beta)$ defined in a domain g.

If $f = (u_1, u_2, u_3)$ and $h = (v_1, v_2, v_3)$ are two vector functions, then

$$(f, h) = \sum_{s=1}^{3} (u_s, v_s)$$

Let us introduce the quadratic functional of the problem (1.1), (1.2)

$$J(f) = \left(\left(\frac{h^2}{12} N + L\right) f, f\right) \tag{2.2}$$

We note that although the operator $(h^2/12)\, N + L$ is elliptic, neither N nor L possess this property. This makes direct utilization of the results obtained in [10, 11] difficult.

Let us prove the following proposition.

Lemma 1. The inequality

$$\begin{gathered} C_1 h^2 \|\partial^2 u_3\|^2 + C_2 \sum_{i=1}^{2} \|\partial u_i\|^2 - C_3 \|f\|^2 \leqslant J(f) \leqslant \\ C_1^* h^2 \|\partial^2 u_3\|^2 + C_2^* \sum_{i=1}^{2} \|\partial u_i\|^2 + C_3^* \|f\|^2 \end{gathered} \tag{2.3}$$

is valid for a smooth vector-function f satisfying the conditions (1.2). Here

$$\begin{gathered} \|\partial u\|^2 = \iint_g \left[\left(\frac{\partial u}{\partial \alpha}\right)^2 + \left(\frac{\partial u}{\partial \beta}\right)^2\right] d\alpha\, d\beta \\ \|\partial^2 u\|^2 = \iint_g \left[\left(\frac{\partial^2 u}{\partial \alpha^2}\right)^2 + \left(\frac{\partial^2 u}{\partial \alpha\, \partial \beta}\right)^2 + \left(\frac{\partial^2 u}{\partial \beta^2}\right)\right] d\alpha\, d\beta \\ \|u\|^2 = \iint_g u^2 d\alpha\, d\beta, \qquad \|f\|^2 = \sum_{i=1}^{3} \|u_i\|^2 \end{gathered} \tag{2.4}$$

C_i, C_i^* $(i = 1, 2, 3)$ are positive constants independent of f and of the parameter h. The proof of the lemma is based on the following relationships, which will be used over and over again later:

$$\begin{gathered} \frac{h^2}{12} (n_{33} u_3, u_3) = \frac{h^2}{12} \iint_g \left\{ \frac{B}{A^3} \left(\frac{\partial^2 u_3}{\partial \alpha^2}\right)^2 + \frac{A}{B^3} \left(\frac{\partial^2 u_3}{\partial \beta^2}\right)^2 + \frac{2}{AB} \left(\frac{\partial^2 u_3}{\partial \alpha\, \partial \beta}\right)^2 + \right. \\ \left. \frac{2\sigma}{AB} \left[\frac{\partial^2 u_3}{\partial \alpha^2} \frac{\partial^2 u_3}{\partial \beta^2} - \left(\frac{\partial^2 u_3}{\partial \alpha\, \partial \beta}\right)^2\right] \right\} d\alpha\, d\beta + h^2 O(\varepsilon \|\partial^2 u_3\|^2 + \varepsilon^{-3} \|u_3\|^2) \end{gathered} \tag{2.5}$$

The first formula (2.5) and all those presented below are valid for all sufficiently small ε. The constants in the O-terms depend only on the coefficients of the system

$$\begin{aligned}
&(l_{11}u_1, u_1) = \iint_g \left[\frac{B}{A}\left(\frac{\partial u_1}{\partial \alpha}\right)^2 + \frac{1-\sigma}{2}\,\frac{A}{B}\left(\frac{\partial u_1}{\partial \beta}\right)^2\right] d\alpha\, d\beta + \\
&\quad O(\varepsilon\|\partial u_1\|^2 + \varepsilon^{-1}\|u_1\|^2) \\
&(l_{12}u_2, u_1) = \iint_g \left[\sigma\,\frac{\partial u_1}{\partial \alpha}\,\frac{\partial u_2}{\partial \beta} + \frac{1-\sigma}{2}\,\frac{\partial u_1}{\partial \beta}\,\frac{\partial u_2}{\partial \alpha}\right] d\alpha\, d\beta + \\
&\quad O\left[\varepsilon\sum_{i=1}^{2}\|\partial u_i\|^2 + \varepsilon^{-1}\sum_{i=1}^{2}\|u_i\|^2\right] \\
&(l_{21}u_1, u_2) = (l_{12}u_2, u_1) \\
&(l_{22}u_2, u_2) = \iint_g \left[\frac{A}{B}\left(\frac{\partial u_2}{\partial \beta}\right)^2 + \frac{1-\sigma}{2}\,\frac{B}{A}\left(\frac{\partial u_2}{\partial \alpha}\right)^2\right] d\alpha\, d\beta + \\
&\quad O(\varepsilon\|\partial u_2\|^2 + \varepsilon^{-1}\|u_2\|^2)
\end{aligned} \tag{2.6}$$

For $(l_{i3}u_3, u_i) = (l_{3i}u_i, u_3)$ $(i = 1, 2, 3)$, estimates of two kinds are needed

Rough

$$\begin{aligned}
&(l_{3i}u_i, u_3) = O(\varepsilon\|\partial u_i\|^2 + \varepsilon^{-1}\|f\|^2) \quad (i=1,2) \\
&(l_{33}u_3, u_3) = O(\|u_3\|^2)
\end{aligned} \tag{2.7}$$

More exact

$$\begin{aligned}
&(l_{31}u_1, u_3) = -\iint_g B(R_1^{-1} + \sigma R_2^{-1})\frac{\partial u_1}{\partial \alpha}\, u_3\, d\alpha\, d\beta + O(\varepsilon\|u_3\|^2 + \varepsilon^{-1}\|u_1\|^2) \\
&(l_{32}u_2, u_3) = -\iint_g A(\sigma R_1^{-1} + R_2^{-1})\frac{\partial u_2}{\partial \beta}\, u_3\, d\alpha\, d\beta + O(\varepsilon\|u_3\|^2 + \varepsilon^{-1}\|u_2\|^2) \\
&(l_{33}u_3, u_3) = \iint_g AB(R_1^{-2} + 2\sigma R_1^{-1}R_2^{-1} + R_2^{-2})u_3^2\, d\alpha\, d\beta
\end{aligned} \tag{2.8}$$

Formulas (2.5) – (2.7) and the first two formulas in (2.8) are proved by integration by parts taking account of the boundary conditions (1.2). The Cauchy-Buniakowski and Sobolev inequalities ([12], p.119, [13], p.96) are hence used

$$\begin{aligned}
&\iint_g \left|\frac{\partial^2 u}{\partial \alpha\,\partial \beta}\,\frac{\partial u}{\partial \alpha}\right| d\alpha\, d\beta \leqslant \varepsilon\|\partial^2 u\|^2 + \varepsilon^{-1}\|\partial u\|^2 \\
&\iint_g \left(\frac{\partial u}{\partial \alpha}\right)^2 d\alpha\, d\beta \leqslant \varepsilon^2\|\partial^2 u\|^2 + \varepsilon^{-2}\|u\|^2
\end{aligned} \tag{2.9}$$

We note also that the estimate

$$\sum_{i,j=1}^{2}(l_{ij}u_j, u_i) \geqslant \frac{1-\sigma}{2}\sum_{i=1}^{2}(l_{ii}u_i, u_i) + O\left(\varepsilon\sum_{i=1}^{2}\|\partial u_i\|^2 + \varepsilon^{-1}\sum_{i=1}^{2}\|u_i\|^2\right) \tag{2.10}$$

follows from (2.6). Taking account of (2.10) and (2.2), formulas (2.5) – (2.8) result in the left inequality of (2.3). The right inequality of (2.3) is obvious. Lemma 1 is proved. By virtue of known theorems (see [12], p. 83, [13], p. 295) it follows at once from (2.3) that the problem (1.1), (1.2) has a discrete spectrum. The positivity of the spectrum results from the formula

$$J(f) = \iint_g AB\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\sigma)\left(\varepsilon_1\varepsilon_2 - \frac{\omega^2}{4}\right)\right] d\alpha\, d\beta + \frac{h^2}{12}\iint_g AB\,[(\varkappa_1+\varkappa_2)^2 - 2(1-\sigma)(\varkappa_1\varkappa_2 - \tau^2)]\, d\alpha\, d\beta \tag{2.11}$$

Here (see [2], p. 299)

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u_1}{\partial u} + \frac{1}{AB}\frac{\partial A}{\partial \beta}u_2 - R_1^{-1}u_3$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial u_2}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u_1 - R_2^{-1}u_3$$

$$\omega = \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u_1}{A}\right) + \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{u_2}{B}\right)$$

$$\varkappa_1 = \frac{1}{A^2}\frac{\partial^2 u_3}{\partial\alpha^2} - \frac{1}{A^3}\frac{\partial A}{\partial\alpha}\frac{\partial u_3}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial u_3}{\partial\beta}$$

$$\varkappa_2 = \frac{1}{B^2}\frac{\partial^2 u_3}{\partial\beta^2} - \frac{1}{B^3}\frac{\partial B}{\partial\beta}\frac{\partial u_3}{\partial\beta} + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial u_3}{\partial\alpha}$$

$$\tau = \frac{1}{AB}\frac{\partial^2 u_3}{\partial\alpha\,\partial\beta} - \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial u_3}{\partial\alpha} - \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial u_3}{\partial\beta}$$

The right side of (2.11) differs from the shell strain potential energy by the factor $^1/_2\, Eh\,(1-\sigma^2)^{-1}$

Lemma 2. The inequality (2.3) is valid for the functional (2.11) for any smooth vector function.

It is sufficient to note for the proof that after removing the parentheses in (2.11) the functional $J(f)$ is represented, for any f, as the sum of the right sides of (2.5), (2.6), (2.8) with the same O-terms. The exception is the inequality (2.10), which we derive by integrating by parts in the second formula in (2.6) by relying on the first two boundary conditions in (1.2). However, this can be avoided if the following Korn type inequality valid for all smooth u_1 and u_2 is used:

$$\iint_g\left[\frac{B}{A}\left(\frac{\partial u_1}{\partial\alpha}\right)^2 + \sigma\frac{\partial u_1}{\partial\alpha}\frac{\partial u_2}{\partial\beta} + \frac{A}{B}\left(\frac{\partial u_2}{\partial\beta}\right)^2 + \frac{1-\sigma}{2}\left(\left(\frac{A}{B}\right)^{1/2}\frac{\partial u_1}{\partial\beta} + \left(\frac{B}{A}\right)^{1/2}\frac{\partial u_2}{\partial\beta}\right)^2\right]d\alpha\, d\beta + \rho\iint_g (u_1^2+u_2^2)\,d\alpha\,d\beta \geqslant \rho'\iint_g\sum_{i=1}^{2}\left[\left(\frac{\partial u_i}{\partial\alpha}\right)^2 + \left(\frac{\partial u_i}{\partial\beta}\right)^2\right]d\alpha\,d\beta$$

Here ρ and ρ' are positive constants (see [14], p. 183).

Henceforth, wherever necessary, we write $J_3(f)$ and $J_c(f)$ depending on whether f satisfies the boundary conditions (1.2) or is an arbitrary smooth vector function in g.

3. Clamped, free and periodic problem in a square. Let us cover the domain g of variation of the parameters α, β with a square mesh with the side Δ. Then we obtain for the distribution function $N(\lambda)$ of the problem (1.1), (1.2) (by analogy with [8])

$$\sum_{(j)} n^{(3,j)}(\lambda) \leqslant N(\lambda) \leqslant \sum_{(j)} n^{(c,j)}(\lambda)$$

Here $n^{(3,j)}(\lambda)$ denotes the distribution function of the eigenvalues of a problem of type (1.1), (1.2) in the cell Q_j, while $n^{(c,j)}(\lambda)$ is the distribution function for the free problem, which consists in seeking the sequence of minima of the functional (2.11) defined in Q_j. Let us clarify that the free problem also has a discrete spectrum by virtue of Lemma 2. The fundamental difficulty consists in seeking the asymptotics for $n^{(3,j)}(\lambda)$ and $n^{(c,j)}(\lambda)$ as $h \to 0$. This problem is solved as follows.

Let Q_j be a square with center at the point (α_j, β_j). Assuming the integrals in the right sides of (2.5), (2.6), (2.8) to be taken with respect to Q_j, let us replace A, B, R_1^{-1}, R_2^{-1} in these integrals by their values at some point $(\alpha_j^*, \beta_j^*) \in Q_j$, we discard the O-terms, and denote by $J_j^\circ(f)$ the sum of the functionals which thus originate. Let us agree to write $J_{3,j}(f)$, $J_{3,j}^\circ(f)$, etc. in the apparent notation. The following assertion is valid.

Lemma 3. For certain positive C and all sufficiently small Δ

$$\overset{\circ}{J}_{3,j}(f)(1 - C\Delta) \leqslant J_{3,j}(f) \leqslant \overset{\circ}{J}_{3,j}(f)(1 + C\Delta) \tag{3.1}$$

$$\overset{\circ}{J}_{c,j}(f)(1 - C\Delta) - C\Delta^{-1}\sum_{i=1}^{2}(u_i, u_i) - Ch^2\Delta^{-3}(u_3, u_3) \leqslant J_{c,j}(f) \leqslant \tag{3.2}$$

$$\overset{\circ}{J}_{c,j}(f)(1 + C\Delta) + C\Delta^{-1}\sum_{i=1}^{2}(u_i, u_i) + Ch^2\Delta^{-3}(u_3, u_3)$$

The estimates (3.1) and (3.2) are obtained if the coefficients B / A^3, $2 / AB$, etc. in the right sides of (2.5), (2.6), (2.8) are replaced by their Taylor series expansions, and $\varepsilon = \Delta$ is set everywhere in the O-terms. In the case of a clamped shell the inequalities

$$(n_{33}^\circ u_3, u_3) \geqslant C\Delta^{-4}(u_3, u_3), \quad (l_{ii}^\circ u_i, u_i) \geqslant C\Delta^{-2}(u_i, u_i), \quad i = 1, 2$$

should also be used.

Now, besides the square Q_j let us consider a square q_i with the same center (α_j, β_j) and side $\Delta - 2\delta$ (Fig. 1). Let $\psi_1(\alpha, \beta)$ and $\psi_2(\alpha, \beta)$ be two infinitely differentiable functions in Q_j such that

$$\psi_1^2(\alpha, \beta) + \psi_2^2(\alpha, \beta) \equiv 1, \qquad (\alpha, \beta) \in Q_j \tag{3.3}$$

$$\psi_2(\alpha, \beta) \equiv 0 \qquad (\alpha, \beta) \in q_j \tag{3.4}$$

where $\psi_1(\alpha, \beta)$ vanishes at some strip Q_j near the boundary which is less than $\delta / 3$. wide. The functions $\psi_i(\alpha, \beta)$ $(i = 1, 2)$ can be chosen such that

$$\left| \frac{\partial^k \psi_i}{\partial\alpha^{k_1}\partial\beta^{k_2}} \right| \leqslant C\delta^{-k}, \qquad k = 1, 2, 3, 4 \tag{3.5}$$

with a constant independent of δ and Δ. The existence of such functions is proved by a standard method (see [15], p. 11, for example). The following assertion later plays an essential part.

Lemma 4. For any smooth vector function $f = (u_1, u_2, u_3)$ with some $C > 0$ and

$$\varkappa = C\delta^{-2}, \qquad \tau = Ch^2\delta^{-4}\Delta^{-1}$$

the inequalities

$$J^{\circ}(f\psi_1) + J^{\circ}(f\psi_2) \leqslant \{J^{\circ}(f) + \varkappa\,[(u_1, u_1) + (u_2, u_2)] + \tau(f, f)\}(1 + C\Delta) \quad (3.6)$$

$$\sum_{i=1}^{2} \{J^{\circ}(f\psi_i) - \varkappa\,[(u_1\psi_i, u_1\psi_i) + (u_2\psi_i, u_2\psi_i)] - \tau\,(f\psi_i, f\psi_i)\} \leqslant J^{\circ}(f)\,(1 + C\Delta) \quad (3.7)$$

are valid. We omit the subscript j here and henceforth below.

The inequality (3.6) is established by direct substitution of the vector-functions $f\psi_1$ and $f\psi_2$ in the functional $J^{\circ}(f)$. The identity (3.3), the estimate (3.5) and inequalities of the type (2.9) should hence be used. The inequality (3.7) is obtained from (3.6) by an obvious transformation taking account of the identity (3.3).

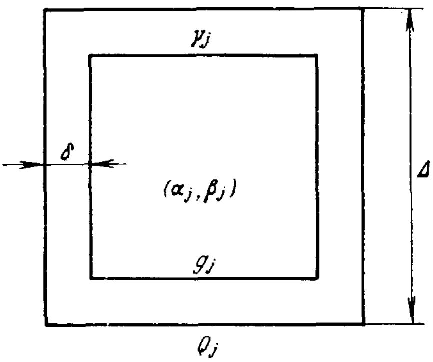

Fig. 1

We note that the $J^{\circ}(f)$ in the right sides in the inequalities (3.6) and (3.7) can be replaced by $J(f)$. This can be done because of the left inequality (3.2) taking account of the evident inequalities $\varkappa > \Delta^{-1}$, $\tau > h^2\Delta^{-3}$. Let us agree to number the inequalities thus obtained (3.6') and (2.7').

Now, let $n^{(3)}(\lambda)$ be the distribution function of the clamped problem on the square Q_j and $n^{(c)}(\lambda)$ the distribution function of minima in the free problem. Let us agree to indicate the passage from the functional $J(f)$ to $J^{\circ}(f)$ by a zero superscript; the bar on top denotes the passage to the functional considered on the borders γ_i, and the pair of subscripts $(\varkappa, \tau)$ the component

$$\varkappa\,[(u_1, u_1) + (u_2, u_2)] + \tau\,(f, f)$$

added to the functional.

The next thing is to obtain a lower bound for the distribution function $n^{(3)}(\lambda)$, and an upper bound for the function $n^{(c)}(\lambda)$ in terms of the effectively calculated distribution function $n^{(n,0)}(\lambda)$ of the periodic problem for the functional $J^{\circ}(f)$ in Q_j. Let us prove a number of propositions.

Lemma 5. For all $\lambda > 0$

$$n_{\varkappa,\tau}^{(n,0)}\,[\lambda\,(1 - C\Delta)] - \bar{n}^{(\cdot,0)}\,[\lambda\,(1 - C\Delta)] \leqslant n^{(3)}(\lambda) \quad (3.8)$$

For the proof we substitute an arbitrary vector function f satisfying the periodicity condition on the sides of the square Q_j into (3.6). The sequence of minima of the functional in the left side of (3.6) can be shifted only to the left if the pair of vector functions $f\psi_1$, $f\psi_2$ is replaced by the pair f_1, f_2, respectively, where f_1 is the vector function satisfying the boundary conditions (1.2), and arbitrary elsewhere, and f_2 is an arbitrary smooth vector function in γ_j which equals zero identically for $(\alpha, \beta) \in q_j$. It is hence naturally assumed that the normalization condition

$$(f_1, f_1) + (f_2, f_2) = (f\psi_1, f\psi_1) + (f\psi_2, f\psi_2) = 1$$

is conserved. Then the inequality (3.7) yields at once

$$n_{\varkappa,\tau}^{(n,0)}[\lambda(1-C\Delta)] \leqslant n^{(3,0)}(\lambda) + \bar{n}^{(c,0)}(\lambda)$$

A lower bound of the type (3.8) hence follows for $n^{(3,0)}(\lambda)$. To complete the proof, the inequality $n^{(3)}(\lambda) \geqslant n^{(3,0)}[\lambda(1-C\Delta)]$ resulting from the right inequality in (3.1) should be used.

Lemma 6. For all $\lambda > 0$

$$n^{(c)}(\lambda) \leqslant n_{-\varkappa,-\tau}^{(n,0)}[\lambda(1+C\Delta)] + \bar{n}_{-\varkappa,-\tau}^{(c,0)}[\lambda(1+C\Delta)]$$

Repeating the reasoning presented in the proof of Lemma 5 in application to the inequality (3.7') (*) and to an arbitrary function, we find

$$n^{(c)}[\lambda(1-C\Delta)] \leqslant n_{-\varkappa,-\tau}^{(3,0)}(\lambda) + \bar{n}_{-\varkappa,-\tau}^{(c,0)}(\lambda) \tag{3.9}$$

Since $n_{-\varkappa,-\tau}^{(n,0)}(\lambda) \geqslant n_{-\varkappa,-\tau}^{(3,0)}(\lambda)$ always, then the lemma is proved.

The following rough upper bound is needed later for the distribution function $n^{(c)}(\lambda)$ of the functional (2.11) defined in some domain g.

Lemma 7.
$$n^{(c)}(\lambda) \leqslant C(\lambda + C_3)\,\mathrm{mes}\, g + \frac{C}{h}\sqrt{\lambda + C_3}\,\mathrm{mes}\, g \tag{3.10}$$

Here the constant C_3 is taken from the left inequality (2.3), and mes g is the area of the domain g. The proof follows directly from the inequality (2.3). The distribution function for the three free functionals on the left in (2.3) are known (see [8]. pp. 374, 390, and also [11]). The following estimate for the function $\bar{n}_{-\varkappa,-\tau}^{(c)}(\lambda)$ of the free problem on the borders γ_j results from (3.10) (see (3.9))

$$\bar{n}_{-\varkappa,-\tau}^{(c)}(\lambda) \leqslant C(\varkappa + \lambda + \tau)\,\Delta\delta + C\sqrt{\lambda + \tau}\,\frac{\Delta\delta}{h} \tag{3.11}$$

Together with the estimate (3.11) Lemmas 5 and 6 reduce the question of the estimation of the distribution functions $n^{(3)}(\lambda)$ and $n^{(c)}(\lambda)$ to the estimation of the distribution function $n_{\varkappa,\tau}^{(n,0)}(\lambda)$ of the periodic problem with constant coefficients in the square Q_j. This problem is solved effectively in exponentials and as is shown below

$$n_{\varkappa,\tau}^{(n,0)}(\lambda) = \frac{\sqrt{3}\Delta^2}{4\pi^2 h} A_j B_j \int_0^{2\pi} \mathrm{Re}\,\sqrt{\bar{\lambda} - \Omega_j(\theta)}\,d\theta + \frac{\Delta^2}{h} O\left[\left(\frac{h^{1/2}}{\delta}\right)^{1/2}\right] + O\left(\frac{\Delta}{\sqrt{h}}\right) \tag{3.12}$$

where

$$\bar{\lambda} = \lambda - \tau, \qquad \tau = Ch^2\delta^{-4}\Delta^{-1} \tag{3.13}$$

and all the functions (α, β) are taken at the point (α_j^*, β_j^*). Let us estimate the function $n_{\varkappa,\tau}^{(n,0)}(\lambda)$.

Because of the evident identity $n_{\varkappa,\tau}^{(n,0)}(\lambda) = n_{\varkappa,0}^{(n,0)}(\lambda - \tau)$ it is possible to be limited to an estimate of the distribution function $n_{\varkappa,0}^{(n,0)}(\lambda)$ of the following problem:

$$\begin{gathered} -\frac{B}{A}\frac{\partial^2 u_1}{\partial\alpha^2} - \frac{1-\sigma}{2}\frac{A}{B}\frac{\partial^2 u_1}{\partial\beta^2} + \varkappa u_1 - \frac{1+\sigma}{2}\frac{\partial^2 u_2}{\partial\alpha\,\partial\beta} + \left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right) B\frac{\partial u_3}{\partial\alpha} = \lambda A B u_1 \\ -\frac{1+\sigma}{2}\frac{\partial^2 u_1}{\partial\alpha\,\partial\beta} - \frac{A}{B}\frac{\partial^2 u_2}{\partial\beta^2} - \frac{1-\sigma}{2}\frac{B}{A}\frac{\partial^2 u_2}{\partial\alpha^2} + \varkappa u_2 + \left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right) A\frac{\partial u_3}{\partial\beta} = \lambda A B u_2 \end{gathered} \tag{3.14}$$

---

*) Let us recall that (3.7') is obtained from (3.7) by replacing $J^\circ(f)$ by $J(f)$ in the right side of (3.7).

$$-\left(\frac{1}{R_1}+\frac{\sigma}{R_2}\right) B\frac{\partial u_1}{\partial\alpha}-\left(\frac{\sigma}{R_1}+\frac{1}{R_2}\right) A\frac{\partial u_2}{\partial\beta}+AB\left(\frac{1}{R_1^2}+\frac{2\sigma}{R_1R_2}+\frac{1}{R_2^2}\right) u_3+\mu^4\frac{1}{AB}\left(\frac{B}{A}\frac{\partial^2}{\partial\alpha^2}+\frac{A}{B}\frac{\partial^2}{\partial\beta^2}\right)\left(\frac{B}{A}\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{A}{B}\frac{\partial^2 u_3}{\partial\beta^2}\right)=\lambda ABu_3\left(\mu^4=\frac{h^2}{12}\right)$$

with periodic conditions on the sides of the square Q_j. Here A, B, R_1, R_2 are values of the corresponding functions at the point $(\alpha_j^*, \beta_j^*)\in Q_j$. The system (3.14) is a Lagrange-Euler system for the functional $J_{\varkappa,0}^{\nu}(f)$. It can be shown that all the eigenvector-functions of the problem (3.14) are

$$f_{k_1,k_2,s}\exp\left[\frac{2\pi i}{\Delta}(k_1\alpha+k_2\beta)\right]$$

$$k_1, k_2=0, \pm 1, \pm 2,\ldots,\quad s=1, 2, 3$$

where $f_{k_1,k_2,s}$ are constant vectors and the corresponding eigenvalues $\lambda_{k_1,k_2,s}$ are roots of the equation

$$D_{k_1k_2}(\lambda)\equiv\det\|d_{ij}(\lambda)\|_{i,j=1}^{3}=0 \tag{3.15}$$

Here

$$d_{11}=\frac{B}{A}\left(\frac{2\pi k_1}{\Delta}\right)^2+\frac{1-\sigma}{2}\frac{A}{B}\left(\frac{2\pi k_2}{\Delta}\right)^2+\varkappa-\lambda AB$$

$$d_{12}=d_{21}=\frac{1+\sigma}{2}\left(\frac{2\pi}{\Delta}\right)^2 k_1k_2$$

$$d_{13}=\bar d_{31}=(R_1^{-1}+\sigma R_2^{-1})B\left(\frac{2\pi i}{\Delta}\right)k_1$$

$$d_{22}=\frac{A}{B}\left(\frac{2\pi k_2}{\Delta}\right)^2+\frac{1-\sigma}{2}\frac{B}{A}\left(\frac{2\pi k_1}{\Delta}\right)^2+\varkappa-\lambda AB$$

$$d_{23}=\bar d_{32}=(\sigma R_1^{-1}+R_2^{-1})A\left(\frac{2\pi i}{\Delta}\right)k_2$$

$$d_{33}=\frac{\mu^4}{AB}\left[\frac{B}{A}\left(\frac{2\pi k_1}{\Delta}\right)^2+\frac{A}{B}\left(\frac{2\pi k_2}{\Delta}\right)^2\right]^2+(R_1^{-2}+2\sigma R_1^{-1}R_2^{-1}+R_2^{-2}-\lambda)AB$$

It is easy to see that $n_{\varkappa,0}^{(n,0)}$ equals the number of integer points (k_1, k_2) for which (3.15) has roots not exceeding a given λ (each pair k_1, k_2 is considered as many times as it corresponds to eigenvalues). Let us substitute

$$\frac{2\pi}{A\Delta}k_1=r\cos\theta,\qquad \frac{2\pi}{B\Delta}k_2=r\sin\theta$$

in (3.15) (cf [3, 6]). Expanding $D_{k_1k_2}(\lambda)$ in decreasing powers of r, we obtain

$$D_{k_1k_2}(\lambda)\equiv\mu^4r^8+A_{0,j}(\lambda)\mu^4r^6+\frac{2}{1-\sigma}(\varkappa-\lambda A_jB_j)^2\mu^4r^4-A_{1,j}(\theta,\lambda)r^4+A_{2,j}(\theta,\lambda)r^2+A_{3,j}(\lambda)=0 \tag{3.16}$$

Here

$$A_{1,j}(\theta,\lambda)=\lambda-(1-\sigma^2)\left(\frac{\sin^2\theta}{R_{1,j}}+\frac{\cos^2\theta}{R_{2,j}}\right)^2$$

and $A_{0,j}, A_{2,j}, A_{3,j}$ are some uniformly bounded functions in θ, j and $0\leqslant\lambda\leqslant\lambda_0$ whose explicit form is not essential henceforth. Substituting $r=\rho/\mu$, we reduce (3.16) to

$$\rho^8-A_{1,j}(\theta,\lambda)\rho^4+\rho^2 O\left[\left(\frac{\mu}{\delta}\right)^2\right]+O\left[\left(\frac{\mu}{\delta}\right)^4\right]=0 \tag{3.17}$$

Equation (3.17) has been studied in detail in [6] in connection with the frequency distribution in shells of revolution. Repeating the reasoning, we find (see [6], formula (3.31))

$$n_{\varkappa,0}^{(n,0)}(\lambda) = \frac{\Delta^2 A_j B_j}{8\pi^2\mu^2}\int_0^{2\pi} \operatorname{Re} A_{1,j}^{1/2}(\theta,\lambda)\,d\theta + O\left(\frac{\Delta}{\mu}\right) + \frac{\Delta^2}{\mu^2}O\left[\left(\frac{\mu}{\delta}\right)^{1/2}\right] \tag{3.18}$$

under the assumption that λ belongs to the set of values of the function $\Omega(\theta, \alpha, \beta)$ (see (1.6)). If λ lies above this set, then the estimate of the accuracy in (3.18) is improved. We omit corresponding results here.

4. Proof of the asymptotic formula. Using Lemmas 5 and 6 and the estimates (3.12) and (3.11), we obtain after summing in the right and left sides of the inequality at the beginning of Sect. 3

$$\begin{aligned} N(\lambda) = \frac{\sqrt{3}}{4\pi^2 h}\Bigg\{ & \iint_g \left(\int_0^{2\pi} \operatorname{Re}\sqrt{\lambda - \Omega(\theta,\alpha,\beta)}\,d\theta\right) AB\,d\alpha\,d\beta + \\ & O\left[\left(\frac{h^{1/2}}{\delta}\right)^{1/2}\right] + O\left(\frac{h^{1/2}}{\Delta}\right) + O(\Delta) + O(h^2\delta^{-4}\Delta^{-1}) + \\ & O\left(\frac{\delta}{\Delta}\right) + O\left(\frac{h}{\Delta\delta}\right) + O\left(\frac{h}{\Delta}\right) + O(\Delta)\Bigg\} \end{aligned} \tag{4.1}$$

The sum of the principal terms in (3.12) yields the integral in (4.1) after substitution of λ for $\lambda(1 \pm C\Delta) + \tau$. The residuals originating here are included in the third and fourth O-terms in the right side of (4.1). It is hence assume that

$$I(\lambda) = \iint_g \left(\int_0^{2\pi} \operatorname{Re}(\lambda - \Omega(\theta,\alpha,\beta))^{-1/2}\,d\theta\right) AB\,d\alpha\,d\beta < +\infty \tag{4.2}$$

The first and second O-terms in (4.1) were obtained as a result of summing the O-terms from (3.12) over the inner squares. The fifth and sixth originated in the addition of the functions $\bar{n}^{(c,j)}(\lambda)$ in (3.11). The contributions of the cells near the boundary (3.10) estimate the seventh and eighth. Setting $\delta = h^{1/4}$ and $\Delta = h^{1/8}$ in (4.1), we arrive at formula (1.5) for $\gamma = 1/8$. If the integral $I(\lambda)$ diverges for given λ, then reasoning analogously, we obtain the O-term in (1.5) with $\gamma = 1/10$. The theorem from Sect. 1 is proved.

So rough an estimate of the reminder is associated with the generality of the problem and the specifics of the method. In fact, the order of the remainder term is substantially higher, however, this is proved successfully only in particular cases (see Sect. 6).

Now, let us explain that if the assumption (1.4) is discarded and n_{ij} are replaced by operators described in Note 1, then (1.5) does not change. The fact is that all the estimates in (2.5) – (2.8) are retained under this substitution, and therefore, so is all the reasoninig. Formula (1.5) thereby turns out to be valid if we proceed from the relationships between the stresses and strains presented in [16], say, and write the self-adjoint system in displacements by eliminating the stresses and moments by means of the method indicated in [1].

Let us note that (1.5) is valid for the system (1.1) for any boundary conditions if only the quadratic functional of the problem agrees with the functional (2.11). The proof is retained completely.

5. Density of distribution of the natural frequencies. Let us set $\lambda = (1 - \sigma^2)\rho E^{-1}\omega^2$ in (1.5), where ω is a frequency parameter. Fixing ω_0 and $\Delta\omega$, we consider the ratio (mean frequency density)

$$\frac{\Delta N}{\Delta \omega} = \frac{N(\omega_0 + \Delta\omega) - N(\omega_0)}{\Delta \omega}$$

Using (1.5) and the Lagrange formula, we obtain

$$\frac{\Delta N}{\Delta \omega} = \frac{(1-\sigma^2)\rho\sqrt{3}}{4\pi^2 Eh}\left\{\iint\limits_g \left[\int\limits_0^{2\pi} \omega_* \operatorname{Re}\left(\frac{(1-\sigma^2)\rho}{E}\omega_*^2 - \Omega(\theta, \alpha, \beta)\right)^{-1/2} d\theta\right] \times \right. \quad (5.1)$$

$$\left. AB\, d\alpha\, d\beta + \frac{O(h^\gamma)}{\Delta\omega}\right\}, \qquad \omega_0 < \omega_* < \omega_0 + \Delta\omega$$

Under the assumption of the smallness of $(\Delta\omega)^{-1} O(h^\gamma)$ formula (5.1) naturally reduces to the consideration of the function

$$I(\omega) = \iint\limits_g \left[\int\limits_0^{2\pi} \omega \operatorname{Re}\left(\frac{(1-\sigma^2)\rho}{E}\omega^2 - \Omega(\theta, \alpha, \beta)\right)^{-1/2} d\theta\right] AB\, d\alpha\, d\beta \qquad (5.2)$$

If the integral (5.2) converges in the neighborhood of the point ω_0, then the location of ω_* in the segment $[\omega_0, \omega_0 + \Delta\omega]$ is hardly essential, and the graph of the function (5.2) reliably characterizes the behavior of the density. The divergence of the integral $I(\omega_0)$ understandably indicates an increase in the density in the neighborhood of ω_0, however, the uncertainty in the location of ω_* in this case makes quantitative estimates difficult. In connection with the above, we note that the integral (5.2) can diverge only in the following three cases

a) $R_1^{-1}(\alpha_0, \beta_0) = R_2^{-1}(\alpha_0, \beta_0)$

$$\frac{\partial}{\partial\alpha}(R_2^{-1} - R_1^{-1})|_0 = \frac{\partial}{\partial\beta}(R_2^{-1} - R_1^{-1})|_0 = 0 \text{ and } \omega_0^2 = \frac{E}{\rho} R_1^{-2}(\alpha_0, \beta_0)$$

b) $R_1(\alpha, \beta) \equiv \text{const}$ in some neighborhood of the point (α_0, β_0) and $\omega_0^2 = (E/\rho) R_1^{-2}(\alpha_0, \beta_0)$;

c) $R_2(\alpha, \beta) \equiv \text{const}$ in some neighborhood of the point (α_0, β_0) and $\omega_0^2 = (E/\rho) R_1^{-2}(\alpha_0, \beta_0)$.

To simplify the formulation we here consider $R_1^{-1}(\alpha, \beta)$ and $R_2^{-1}(\alpha, \beta)$ to be analytic in the neighborhood of (α_0, β_0).

We present graphs of the function $I^*(\omega) = I(\omega)/2\pi$ under diverse assumptions relative to the shell geometry. The computations were carried out on a BESM-3M computer. The curves $I^*(\omega)$ are pictured in Fig. 2 for the case of a truncated circular cone with the meridian

$$y = x \operatorname{tg}\alpha + (\sqrt{2} - \sqrt{2}\operatorname{tg}\alpha), \qquad \sqrt{2} - \cos\alpha \leqslant x \leqslant \sqrt{2}$$

(rotation is performed around the x-axis, generator is of unit length throughout). Curves *1 — 3* correspond to values of the slope of the generator $\alpha = {}^1/_8\pi, {}^1/_4\pi, {}^3/_8\pi$. Curve *4* corresponds to a cylinder $\alpha = 0$. The curves $I^*(\omega)$ corresponding to shells of revolution of negative Gaussian curvature are shown in Fig. 3

$$y = 1 + \alpha(1 - x^2), \qquad 0 \leqslant x \leqslant 1$$

Curves *1 — 5* correspond to the values $\alpha = -0.75, -0.5, -0.25, -0.1, 0$. The curves $I^*(\omega)$ for shells of revolution of positive Gaussian curvature with the meridian

$$y = 0.25 + \alpha (1 - x^2), \qquad 0 \leqslant x \leqslant 1$$

are pictured in Fig. 4. Curves *1* — *4* correspond to the values $\alpha = 0,\ 0.1,\ 0.25,\ 0.5$. The curves $I\ (\omega)$ are pictured in Fig. 5 for the hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \qquad x^2 + y^2 \leqslant 1$$

Curves *1* — *3* correspond to the values $b^2 = 3,\ 4,\ 5,\ a^2 = {}^1/_2\, b^2$. It is assumed everywhere that $\sigma = 0.3$. Moreover, also $\rho\ (1 - \sigma^2)\ E^{-1} = 1$. Other values of ρ and E will evidently result in a change in scale along the axes. Points correspond-

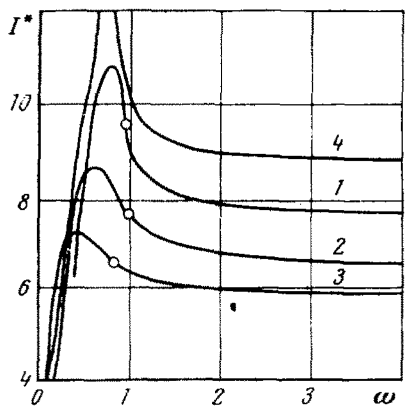

Fig. 2

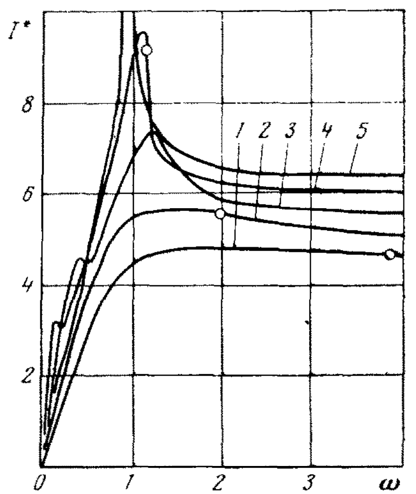

Fig. 3

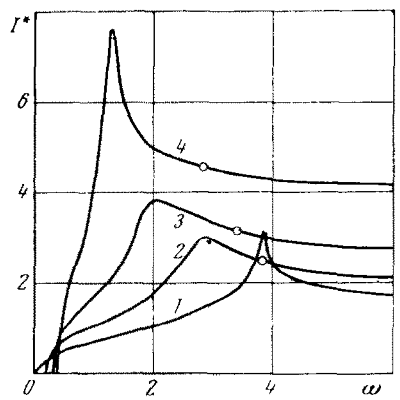

Fig. 4

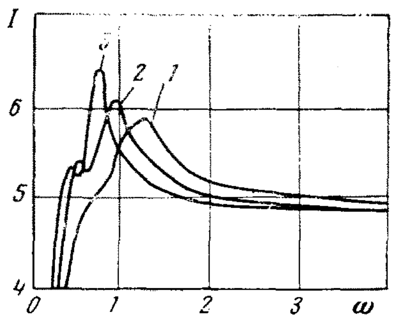

Fig. 5

ing to the ends of the segments of a set of values of the function $\sqrt{\Omega\ (\theta,\ \alpha,\ \beta)}$ are noted on the graphs. Attention is turned to the influence of the limit spectrum of the membrane problem on the value of the density.

In conclusion, let us note that as follows from (5.2), the curve $I\ (\omega)$ can, in principle, have any number of extrema. Examples of this kind can be constructed for shells of both positive and negative Gaussian curvature.

6. Certain remarks. 1°. In the case of shells of revolution (1.5) becomes

$$N(\lambda) = \frac{\sqrt{3}}{2\pi h}\left[\int_a^b B(s)\left(\int_0^{2\pi} \operatorname{Re} \sqrt{\lambda - \Omega(\theta, s)}\, d\theta\right) ds + O(h^\gamma)\right] \qquad (6.1)$$

where $B(s)$ $(a \leqslant s \leqslant b)$ is the distance between a point on the shell and the axis of revolution, and s is the arclength of the meridian. The reminder term in (6.1) is obtained in [6] with $\gamma = 1/4$ if (4.2) is satisfied and with $\gamma = 1/6$ without this assumption. This is also valid for the remainder term in the case of an arbitrary cylindrical shell.

2°. Setting $h = 0$ in (1.1) and discarding the last two boundary conditions in (1.2), we obtain the membrane problem (see [17]).

As shown in [18], in the case of shells of revolution the set of values of the function $\Omega(\theta, s)$ $(s \in [a, b], \theta \in [0, 2\pi])$ agrees with the limit spectra (see the definition in [19], p. 316) of the membrane problem to the accuracy of two isolated points. This circumstance relates (6.1) to the Bohr formula from quantum mechanics (see [20]), where a function whose values generate the continuous spectrum of the degenerate problem is also under the integral sign for the distribution function. Apparently the distribution function has such a form every time the spectrum of the degenerate operator contains continuous pieces. In this connection, let us mention the assumption that (1.5) behaves stably with respect to the selection of the perturbing operator N. Complementing what has been mentioned in Sect. 4, it can be shown that it does not vary if arbitrary fourth order operators are taken as n_{ij} $(i, j \leqslant 2)$ with the sole constraint that the operator N be positive-definite.

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SOME DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY

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On the basis of the functionally-invariant solutions of the wave equation, suggested by Smirnov and Sobolev, we give a closed solution of a class of selfsimilar problems of the dynamical theory of elasticity. This class contains the following problems: (a) a half-plane, arbitrarily loaded at the boundary (including the case when the endpoints of the loaded segments move with arbitrary constant velocities); (b) the contact problem for the half-plane, when the ends of the contact areas are displaced with arbitrary constant velocities; (c) a collection of arbitrarily loaded cuts along the same line, moving with constant velocities, the different endpoints of the cuts having, possibly, different velocities. The solution of the indicated problems are reduced in the simplest cases to the Dirichlet problem or to the mixed Keldysh-Sedov problems of the theory of analytic functions of a complex variable. In principle, the procedure for finding the solution